SOLUTIONS

UBC Math 104/184 Exams

April 2013 Exam Solutions

1. (a) $\lim_{x\to 2}\frac{\sqrt{x^2+12}-x}{x-1} = \frac{\lim_{x\to 2}(\sqrt{x^2+12}-x)}{\lim_{x\to 2}(x-1)} = \frac{\lim_{x\to 2}\sqrt{x^2+12}-\lim_{x\to 2}x}{\lim_{x\to 2}x-\lim_{x\to 2}1} = \frac{\sqrt{16}-2}{2-1} = \frac{4-2}{1} = 2$

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1. (b) Since $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-}\sqrt{x^2+2} = \sqrt{2}$ and $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}(x+1) = 1$,

therefore $\lim_{x\to 0^+} g(x) \neq \lim_{x\to 0^+} g(x)$,

so $\lim_{x\to 0} g(x)$ does not exist. Thus $g(x)$ is not continuous at $x=0$, no matter what the value of a. There is no value of a for which the function $g(x)$ will be continuous everywhere.

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1. (c) By using the product rule, we have

$$\frac{dy}{dx} = x^x\frac{d}{dx}(\ln x)+\ln x\frac{d}{dx}(x^x).$$

In order to find the derivative of x^x, we need to use logarithmic differentiation.

Let $f(x)=x^x$. Then $\ln[f(x)] = \ln[x^x] = x\ln x$, so

$$\frac{d}{dx}(\ln[f(x)]) = \frac{d}{dx}(x\ln x),$$

$$\frac{1}{f(x)}f'(x) = x\frac{d}{dx}(\ln x)+\ln x\frac{d}{dx}(x) = x\cdot\frac{1}{x}+\ln x\cdot 1 = 1+\ln x,$$

$$f'(x) = f(x)\cdot(1+\ln x) = x^x(1+\ln x).$$

So $\frac{d}{dx}(x^x) = x^x(1+\ln x)$, and therefore

$$\frac{dy}{dx} = x^x\frac{d}{dx}(\ln x)+\ln x\frac{d}{dx}(x^x) = x^x\cdot\frac{1}{x}+\ln x\cdot[x^x(1+\ln x)] = x^x\cdot\left(\frac{1}{x}+\ln x(1+\ln x)\right)$$

$$= x^x\cdot\left(\frac{1}{x}+\ln x+\ln^2 x\right).$$

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April 2013 Exam Solutions (cont'd)

1. (d) $\dfrac{d}{dx}(x^2y + xy^2) = \dfrac{d}{dx}(2) \quad \Rightarrow \quad \left(x^2\dfrac{dy}{dx} + y\cdot 2x\right) + \left(x\cdot 2y\dfrac{dy}{dx} + y^2\cdot 1\right) = 0$

Plugging in $(x,y) = (1,1)$ gives

$$\left(1^2\frac{dy}{dx} + 2\right) + \left(2\frac{dy}{dx} + 1^2\right) = 0 \quad \Rightarrow \quad 3\frac{dy}{dx} + 3 = 0 \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{(1,1)} = -1$$

So the slope of the tangent line is $m = -1$. Its equation is therefore

$$y - 1 = -1(x-1) \quad \text{or} \quad y = -x + 2\,.$$

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1. (e) Let $f(x) = x^3$ and $a = 1$. Then $f'(x) = 3x^2$. The linear approximation formula is

$$f(x) \approx f(a) + f'(a)(x-a) = f(1) + f'(1)(x-1) = 1 + 3(x-1)\,.$$

So $\quad (0.998)^3 = f(0.998) \approx 1 + 3(0.998 - 1) = 1 + 3(-0.002) = 1 - 0.006 = 0.994\,.$

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1. (f) As $x \to -1^-$, $x^2 - 4 \to -3$ and $x + 1 \to 0^-$, so $\displaystyle\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}\frac{x^2-4}{x+1} = +\infty\,.$

Similarly, as $x \to -1^+$, $x^2 - 4 \to -3$ and $x + 1 \to 0^+$, so $\displaystyle\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}\frac{x^2-4}{x+1} = -\infty\,.$

Since $\displaystyle\lim_{x\to -1^-} f(x) = +\infty$ and $\displaystyle\lim_{x\to -1^+} f(x) = -\infty$, therefore $x = -1$ is a vertical asymptote.

Also, since the degree of the numerator is 2 and the degree of the denominator is 1, the degree of the numerator is <u>exactly</u> one greater then the degree of the denominator, so there will be a slant asymptote, which can be found by dividing the numerator by the denominator.

$$f(x) = \frac{x^2-4}{x+1} = \frac{x^2-1}{x+1} - \frac{3}{x+1} = \frac{(x+1)(x-1)}{x+1} - \frac{3}{x+1} = (x-1) - \frac{3}{x+1}\,.$$

So $y = x - 1$ is a slant asymptote, and $x = -1$ is a vertical asymptote.

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1. (g) $h'(x) = x^{-2}\dfrac{d}{dx}(e^x) + e^x\dfrac{d}{dx}(x^{-2}) = x^{-2}\cdot e^x + e^x(-2x^{-3}) = e^x(x^{-2}-2x^{-3})$

$$= e^x\left(\frac{1}{x^2}-\frac{2}{x^3}\right) = e^x\left(\frac{x}{x^3}-\frac{2}{x^3}\right) = \frac{x-2}{x^3}e^x$$

At a critical point, $h'(x)=0$, so

$$\frac{x-2}{x^3}e^x = 0 \Rightarrow \frac{x-2}{x^3} = 0 \Rightarrow x-2=0 \Rightarrow x=2.$$

Also $h(2)=2^{-2}e^2=\dfrac{e^2}{2^2}=\frac{1}{4}e^2$, so the point $(2,h(2))=(2,\frac{1}{4}e^2)$ is a critical point.

Since $h'(x)<0$ for $x<2$ and $h'(x)>0$ for $x>2$, the critical point $(2,\frac{1}{4}e^2)$ is a local minimum (by the 1st Derivative Test). There is no local maximum (although the endpoint $(4,h(4))=(4,\frac{1}{16}e^4)$ is an absolute maximum).

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1. (h) $f'(x) = e^x\dfrac{d}{dx}(\sin x)+\sin x\dfrac{d}{dx}(e^x) = e^x\cos x+\sin x\cdot e^x = e^x(\cos x+\sin x)$

$$f''(x) = e^x\frac{d}{dx}(\cos x+\sin x)+(\cos x+\sin x)\frac{d}{dx}(e^x) = e^x(-\sin x+\cos x)+(\cos x+\sin x)e^x$$

$$= e^x[(-\sin x+\cos x)+(\cos x+\sin x)] = 2e^x\cos x$$

The function $f(x)=e^x\sin x$ will be concave upward when $f''(x)=2e^x\cos x>0$. Since $e^x>0$ for all values of *x*, this will happen when $\cos x>0$ (which occurs in the first quadrant, where $0<x<\frac{\pi}{2}$). So the function $f(x)=e^x\sin x$ is concave upward on the interval $[0,\frac{\pi}{2}]$.

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2. (a) (i) $g(0)$ is defined since $g(0)=4\cdot 0^2+2\cdot 0+1=1$.

(ii) $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-}(4x^2+2x+1) = 1$ and $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}\sqrt{4x+1} = \sqrt{1} = 1$.

Since $\lim_{x\to 0^-} g(x)=\lim_{x\to 0^+} g(x)=1$, therefore $\lim_{x\to 0} g(x)=1$. So $\lim_{x\to 0} g(x)$ exists.

(iii) Since $g(0)=1$ and $\lim_{x\to 0} g(x)=1$, therefore $\lim_{x\to 0} g(x)=g(0)$.

So the function $g(x)$ is continuous at $x=0$.

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2. (b) $g'(0)$ is defined by $g'(0) = \lim_{x\to 0}\frac{g(x)-g(0)}{x-0} = \lim_{x\to 0}\frac{g(x)-1}{x}$, if this limit exists. This limit will exist if and only if the left- and right-hand limits both exist and are equal. So we need to check these two limits.

$$\lim_{x\to 0^-}\frac{g(x)-1}{x} = \lim_{x\to 0^-}\frac{(4x^2+2x+1)-1}{x} = \lim_{x\to 0^-}\frac{4x^2+2x}{x} = \lim_{x\to 0^-}(4x+2) = 2,$$

$$\lim_{x\to 0^+}\frac{g(x)-1}{x} = \lim_{x\to 0^+}\frac{\sqrt{4x+1}-1}{x} = \lim_{x\to 0^+}\frac{\sqrt{4x+1}-1}{x}\cdot\frac{\sqrt{4x+1}+1}{\sqrt{4x+1}+1} = \lim_{x\to 0^+}\frac{(4x+1)-1}{x\,(\sqrt{4x+1}+1)}$$

$$= \lim_{x\to 0^+}\frac{4x}{x\,(\sqrt{4x+1}+1)} = \lim_{x\to 0^+}\frac{4}{\sqrt{4x+1}+1} = \frac{4}{\sqrt{1}+1} = \frac{4}{2} = 2.$$

Since $\lim_{x\to 0^-}\frac{g(x)-g(0)}{x-0}=2$ and $\lim_{x\to 0^+}\frac{g(x)-g(0)}{x-0}=2$, the two one-sided limits exist and are equal, so $\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}=2$. Therefore $g'(0)=2$.

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3. (a) The tangent line passes through the points $(1,1)$ and $(2,5)$, so its slope is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1} = \frac{5-1}{2-1} = 4.$$

Since the slope of the tangent line to the graph of $y(x)$ at $(1,1)$ is also given by the value of the derivative $y'(1)$ at this point (and $y' = xy^2+ax^2$), we have

$$y' = 1\cdot 1^2 + a\cdot 1^2 = 1+a.$$

So $1+a=4$, and therefore $a=3$.

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3. (b) Since $a=3$, we have $y' = xy^2+ax^2 = xy^2+3x^2$. Therefore

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(xy^2+3x^2) = \left(x\frac{d}{dx}(y^2)+y^2\frac{d}{dx}(x)\right)+\frac{d}{dx}(3x^2)$$

$$= (x\cdot(2yy')+y^2\cdot 1)+6x = 2xyy'+y^2+6x.$$

At the point $(1,1)$, we have $y'=1+a=4$, so

$$y'' = 2\cdot 1\cdot 1\cdot 4+1^2+6\cdot 1 = 8+1+6 = 15 > 0.$$

So the function is concave upwards at $x=1$.

April 2013 Exam Solutions (cont'd)

4. (a) $\dfrac{dq}{dp} = 1000e^{-p/200} \cdot \dfrac{d}{dp}\left(-\dfrac{p}{200}\right) = 1000e^{-p/200} \cdot \left(-\dfrac{1}{200}\right) = -5e^{-p/200}$.

The elasticity of demand ε is therefore

$$\varepsilon = \frac{p}{q}\cdot\frac{dq}{dp} = \frac{p}{1000e^{-p/200}}\cdot(-5e^{-p/200}) = \frac{-p}{1000}\cdot(5) = -\frac{p}{200} = -0.005p\,.$$

When the price is $p = \$100$ the elasticity of demand is $\varepsilon = -0.005 \cdot 100 = -0.5 = -\frac{1}{2}$.

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4. (b) The revenue is given by $R = pq$, so marginal revenue is

$$MR(p) = \frac{dR}{dp} = \frac{d}{dp}(pq) = p\frac{dq}{dp} + q\frac{dp}{dp} = p\frac{dq}{dp} + q\cdot 1 = q\cdot\left(\frac{p}{q}\frac{dq}{dp}+1\right) = q(\varepsilon+1)\,.$$

When $p = \$100$, $q = 1000e^{-100/200} = 1000e^{-1/2} = \frac{1000}{\sqrt{e}}$ and $\varepsilon = -\frac{1}{2}$, so the marginal revenue is

$$MR(100) = q(\varepsilon+1) = \tfrac{1000}{\sqrt{e}}(-\tfrac{1}{2}+1) = \tfrac{1000}{\sqrt{e}}(\tfrac{1}{2}) = \tfrac{500}{\sqrt{e}}\,.$$

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4. (c) Since the elasticity of demand is $\varepsilon = -\frac{1}{2}$, a 1% increase in the price should cause the demand to drop by $\frac{1}{2}\%$.

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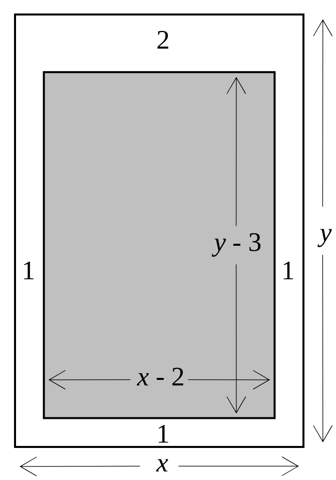
5. Let x and y denote the width and height of the poster, respectively. The width and height of the printed area will then be $x-2$ and $y-3$, respectively. Since the total area of the poster is to be 150 cm^2, we must have $xy = 150$ so $y = \dfrac{150}{x}$. The area of the printed region will then be



$$A = (x-2)(y-3) = xy - 3x - 2y + 6 = 150 - 3x - 2\cdot\frac{150}{x} + 6$$

$$= 156 - 3x - \frac{300}{x} = 156 - 3x - 300x^{-1}\,.$$

Therefore $\dfrac{dA}{dx} = -3 + 300x^{-2} = -3 + \dfrac{300}{x^2}$.

At a critical point, $\dfrac{dA}{dx} = 0$, so

$$-3 + \frac{300}{x^2} = 0 \;\Rightarrow\; 3 = \frac{300}{x^2} \;\Rightarrow\; 3x^2 = 300 \;\Rightarrow\; x^2 = 100 \;\Rightarrow\; x = 10\,.$$

Since $\dfrac{d^2A}{dx^2} = -600x^{-3} = -\dfrac{600}{x^3} = -\dfrac{600}{10^3} < 0$,

$x=10$ gives a local maximum. When $x=10$, $y=\dfrac{150}{x}=\dfrac{150}{10}=15$. So the width and height of the poster should be 10 cm and 15 cm, respectively.

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6. Let $y(t)$ denote the daily sales of tennis rackets at time *t*, where *t* is the number of months since the end of January. We are given that the rate of change of *y* with respect to time *t* is proportional to *y* ($\dfrac{dy}{dt}\propto y$) so $\dfrac{dy}{dt}=ky$ where *k* is some constant of proportionality. The only function with this property is the exponential function $y(t)=Ce^{kt}$ for some constants *C*, *k*.

When $t=0$, $y=1000$, so

$$y(t)=Ce^{kt} \quad\Rightarrow\quad 1000 = Ce^{k\cdot 0} = Ce^{0} = C\cdot 1 = C \quad\Rightarrow\quad C=1000.$$

When $t=2$, $y=600$, so

$$y(t)=1000\,e^{kt} \quad\Rightarrow\quad 600 = 1000\,e^{k\cdot 2} \quad\Rightarrow\quad \tfrac{600}{1000} = e^{2k} \quad\Rightarrow\quad \ln(\tfrac{600}{1000}) = 2k,$$

so $k=\frac{1}{2}\ln(\frac{6}{10})=\frac{1}{2}\ln(0.6)$. Therefore $y(t)=Ce^{kt}=1000\,e^{[\frac{1}{2}\ln(0.6)]\,t}$.

Therefore, in another 4 months (i.e. six months after January), the daily sales will be

$$y(6) = 1000\,e^{[\frac{1}{2}\ln(0.6)]\cdot 6} = 1000\,e^{3\ln(0.6)} = 1000\,(e^{\ln(0.6)})^{3} = 1000\,(0.6)^{3} = 1000\,(0.216) = 216.$$

So the daily sale will be 216 tennis rackets per day in another 4 months.

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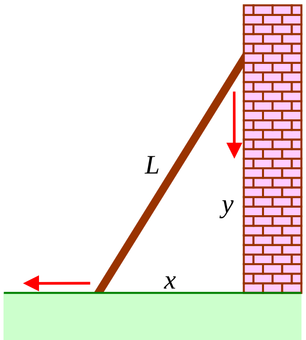
7. Let *L* denote the length of the ladder, and let *x* be the distance of the foot of the ladder from the wall and *y* its height above the ground (see diagram). Then *x* and *y* are variables (functions of time), while *L* is a constant. By the Pythagorean Theorem,



$$x^2+y^2 = L^2.$$

Differentiating with respect to *t* gives

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(L^2) \quad\Rightarrow\quad 2x\frac{dx}{dt}+2y\frac{dy}{dt} = 0.$$

Plugging in $y=4\text{ m}$, $\dfrac{dx}{dt}=+2\text{ m/min}$, $\dfrac{dy}{dt}=-1.5\text{ m/min}$ gives

$$2x\cdot(+2)+2\cdot 4\cdot(-1.5)=0 \quad\Rightarrow\quad 4x-12=0 \quad\Rightarrow\quad x=3.$$

So $\qquad L^2 = x^2+y^2 = 3^2+4^2 = 25 \quad\Rightarrow\quad L=5.$

The length of the ladder is $L=5\text{ m}$.

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8. (a)

$$f(x) = \ln(1+x), \qquad f(0) = \ln(1) = 0;$$

$$f'(x) = \frac{1}{1+x} = (1+x)^{-1}, \qquad f'(0) = 1^{-1} = 1;$$

$$f''(x) = -1(1+x)^{-2}, \qquad f''(0) = -1 \cdot 1^{-2} = -1;$$

$$f'''(x) = +2(1+x)^{-3}, \qquad f'''(0) = 2 \cdot 1^{-3} = 2.$$

The third degree Maclaurin polynomial is

$$T_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 0 + \frac{1}{1}x + \frac{-1}{2}x^2 + \frac{2}{6}x^3 = x - \frac{x^2}{2} + \frac{x^3}{3}.$$

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8. (b) Since $f'''(x) = +2(1+x)^{-3}$,

$$f^{(4)}(x) = -2 \cdot 3(1+x)^{-4},$$

$$f^{(5)}(x) = +2 \cdot 3 \cdot 4(1+x)^{-5}, \qquad \text{etc.}$$

the n^{th} derivative is given by

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n},$$

and the $(n+1)^{\text{th}}$ derivative is given by

$$f^{(n+1)}(x) = (-1)^n n!\,(1+x)^{-(n+1)} = \frac{(-1)^n n!}{(1+x)^{n+1}}.$$

The remainder term is given by $R_n(x) = f(x) - T_n(x) = \dfrac{f^{(n+1)}(t)}{(n+1)!}x^{n+1}$, where t is some number between x and $a = 0$. So

$$R_n(x) = f^{(n+1)}(t)\frac{x^{n+1}}{(n+1)!} = \frac{(-1)^n n!}{(1+t)^{n+1}} \cdot \frac{x^{n+1}}{(n+1)\cdot n!} = \frac{(-1)^n x^{n+1}}{(n+1)(1+t)^{n+1}}.$$

Setting $x = 0.1$ gives

$$R_n(0.1) = f(0.1) - T_n(0.1) = \frac{(-1)^n (0.1)^{n+1}}{(n+1)(1+t)^{n+1}}.$$

where t is some number between $x = 0.1$ and $a = 0$ (i.e. $0 < t < 0.1$). So

$$|R_n(0.1)| = |\ln(1.1) - T_n(0.1)| = \left|\frac{(-1)^n (0.1)^{n+1}}{(n+1)(1+t)^{n+1}}\right| = \frac{(\frac{1}{10})^{n+1}}{(n+1)(1+t)^{n+1}}$$

$$< \frac{(\frac{1}{10})^{n+1}}{(n+1)(1+0)^{n+1}} = \frac{(\frac{1}{10})^{n+1}}{n+1} = \frac{1}{(n+1)(10)^{n+1}}.$$

We can get the error below 10^{-6} by choosing $n=5$ since then

$$\left|R_5(0.1)\right| = \left|\ln(1.1) - T_5(0.1)\right| < \frac{1}{(5+1)(10)^{5+1}} = \frac{1}{6\times10^6} < 10^{-6}.$$

So we should choose the first five terms in the Maclaurin polynomial in order to approximate $\ln(1.1)$ with an error below 10^{-6}.